

0 for $i \geq n$

0 for $i > n$

$$\begin{array}{ccccc} \rightarrow H_i(M, M \setminus \partial U) & \rightarrow & H_i(U \sqcup M \setminus \bar{U}, M \setminus \bar{U}) & \rightarrow & H_i(M, M \setminus \bar{U}) & \rightarrow \\ & & \cong \uparrow \text{excision} & & \uparrow & \\ & & H_i(U) & \xrightarrow{\quad} & H_i(M) & \\ & & [\varphi] & \xrightarrow{\quad} & [\varphi] & \end{array}$$

$\Rightarrow H_i(U) = 0 \quad \forall i > n.$

It remains to deal with the case $i = n$.

Let $x_0 \in M \setminus \bar{U}$ (recall that M is non-cpt. and thus $\emptyset \neq M \setminus \bar{U}$)

$\text{res}_{x_0} [\varphi] = 0$ in $H_n(M, M \setminus x_0)$ since $[\varphi]$ can be regarded as a class in $H_n(M \setminus x_0)$

\Rightarrow the section $M \ni x \mapsto \text{res}_x [\varphi]$ vanishes by connectedness of M .

\Rightarrow the img of $[\varphi]$ in $H_n(M, M \setminus \bar{U}) \cong \Gamma_{\bar{U}} \left(\begin{smallmatrix} M_R \\ \downarrow \\ M \end{smallmatrix} \right)$ vanishes

Looking at the diagram above we thus conclude that $[\varphi] = 0$ in $H_n(U)$.

Torsion freeness follows as before from the UCT.

Cohomology with compact support & Duality for noncpt mfs.

28.05.201

Thm (Poincaré duality) M n -mf. conn. cpt. \mathbb{R} -orientable.

Then $[M] \cap - : H^l(M; \mathbb{R}) \xrightarrow{\cong} H_{n-l}(M; \mathbb{R})$.

For the pf. we need a bit more than just cpts: we need a thg of chg w/ cpt support.

Let M be an n -mf., $K \subseteq M$ cpt subset. Then we have

$H_n(M, M \setminus K; \mathbb{R}) \xrightarrow{\cong} \Gamma_K(M_R), \quad \alpha \mapsto (x \mapsto \text{res}_x(\alpha))_{x \in M}$

where $\Gamma_K(M_R) = \{ \text{c.sections of } M_R \xrightarrow{\text{PR}} M \text{ defined on } K \}$,

$M_R = \coprod_{x \in M} H_n(M, M \setminus x; \mathbb{R})$.

i.e. given a c. section $K \xrightarrow{\text{PR}} M_R \quad \exists! \alpha \in H_n(M, M \setminus K; \mathbb{R})$ s.t. $\forall x \in K, \text{res}_x(\alpha) = \text{c}(x)$.

$$H^l(M, M \setminus K; \mathbb{R}) \stackrel{\text{def}}{=} H^l(C^*(M, M \setminus K; \mathbb{R}))$$

$$\varphi \in C^l(M, M \setminus K; \mathbb{R}) \stackrel{\text{def}}{\iff} \varphi: C_l(M, \mathbb{Z}) \rightarrow \mathbb{R} \text{ \& } \varphi|_{C_l(M \setminus K; \mathbb{Z})} = 0$$

i.e. "functions supported in K "

Ex. $M = \mathbb{R}^n$, $K = x_0$. Consider the collection of functions $\varphi: C_l(\mathbb{R}^n, \mathbb{Z}) \rightarrow \mathbb{Z}$ for which $\varphi(\sigma) = 0$ unless $\text{Im } \sigma \subseteq \{x_0\}$ (i.e. unless $\sigma = \text{const } x_0$)

$$\varphi \in C^0(\mathbb{R}^n, \mathbb{Z}) = \text{Hom}(C_0(\mathbb{R}^n, \mathbb{Z}), \mathbb{Z}). \text{ Let } \sigma_x: \Delta^0 \rightarrow \mathbb{R}^n \quad \forall x \in \mathbb{R}^n$$

$$x \mapsto x$$

$$\varphi(\sigma_{x_0}) = 1 \text{ and } \varphi(\sigma_x) = 0 \quad \forall x \neq x_0.$$

$$\delta\varphi \in C^1(\mathbb{R}^n, \mathbb{Z}), \sigma \in C_1(\mathbb{R}^n, \mathbb{Z}): (\delta\varphi)(\sigma) = \varphi(\sigma(0,1)) - \varphi(\sigma(1,0))$$

Take σ_{xx_0} starting at x_0 and ending at $x \neq x_0$

$$(\delta\varphi)(\sigma_{xx_0}) = 1, \text{ Im } \sigma_{xx_0} \not\subseteq K = \{x_0\}$$

This example shows that the above statement that $C^l(M, M \setminus K)$ is the set of functions supported in K is not quite correct; the actual set of those functions would not be a subcomplex.

Ex. $H_n(\mathbb{R}^n, \mathbb{Z}) = 0 \quad \forall n \geq 1$ Poincaré duality fails:

$$0 = H^n(\mathbb{R}^n, \mathbb{Z}) \xrightarrow[\neq]{0} H_{n-n}(\mathbb{R}^n, \mathbb{Z}) = H_0(\mathbb{R}^n, \mathbb{Z}) = \mathbb{Z}$$

Def. Let X be a space. A compactly supported cochain of dimension l is a $\varphi \in C^l(X; \mathbb{R})$ s.t. $\exists K \subseteq X$ cpt. s.t. $\varphi|_{C_l(X \setminus K; \mathbb{Z})} = 0$, i.e. we have $\varphi \in C^l(X, X \setminus K; \mathbb{R})$. The collection of such cochains is $C_c^l(X; \mathbb{R})$.

Note. Prof. Hutchings does not include T2 in the def. of cptness.

$C_c^*(X; \mathbb{R}) \subseteq C^*(X; \mathbb{R})$ is a sub-cochain complex.

$$\varphi|_{C_l(X \setminus K; \mathbb{Z})} = 0 \implies \delta\varphi|_{C_l(X \setminus K; \mathbb{Z})} = 0, \text{ i.e. } C^*(X, X \setminus K; \mathbb{R}) \text{ is a cochain cx.}$$

$$C_c^*(X; \mathbb{R}) = \bigcup_{\substack{K \subseteq X \\ \text{cpt.}}} \left. \begin{array}{l} C^*(X, X \setminus K; \mathbb{R}) \\ \subseteq C^*(X; \mathbb{R}) \end{array} \right\} \subseteq C^*(X; \mathbb{R})$$

Def. The compactly supported cohomology of X is the cohomology of

$$C_c^*(X; \mathbb{R}) : \quad \underline{H_c^l(X; \mathbb{R})} := H^l(C_c^*(X; \mathbb{R})).$$

How may we compute $H_c^l(X; \mathbb{R})$ in terms of $H^*(X, X \setminus K; \mathbb{R})$?

First we do some algebra: colimits.

Def. Let J be a poset. We call J directed if

$$\forall \alpha, \beta \in J \quad \exists \gamma \in J : \alpha \leq \gamma, \beta \leq \gamma.$$

Def. An J -indexed direct system of ab. groups is a collection

$$\{G_\alpha, f_{\alpha\beta} \mid \alpha \in J, \beta \geq \alpha\} \quad \text{s.t.}$$

- $\forall \alpha \in J$ we have an ab. gp. G_α

- $\forall \alpha, \beta \in J, \alpha \leq \beta$ we have a homomorphism $G_\alpha \xrightarrow{f_{\alpha\beta}} G_\beta$

subject to the conditions

1) $f_{\alpha, \alpha} = \text{id}_{G_\alpha}$

2) $\alpha \leq \beta \leq \gamma \Rightarrow f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$, i.e.

$$G_\alpha \begin{array}{c} \xrightarrow{f_{\alpha\gamma}} \\ \xrightarrow{f_{\alpha\beta}} G_\beta \xrightarrow{f_{\beta\gamma}} \end{array} G_\gamma$$

Note. Any poset can be considered as a category.

objects are the elts of the poset, $\text{Mor}(\alpha, \beta) = \begin{cases} \emptyset & \alpha \not\leq \beta \\ * & \alpha \leq \beta \end{cases}$

Then we may consider J as a category and a direct system is just a functor $J \rightarrow \text{Ab}$.

Ex. An $J = (\mathbb{N}, \leq)$ -directed system on (\mathbb{N}, \leq) is a sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \rightarrow \dots$$

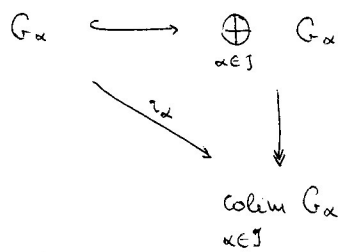
with $f_{nm} = f_{m-1} \circ \dots \circ f_n : A_n \xrightarrow{f_n} \dots \xrightarrow{f_{m-1}} A_m \quad \forall n \leq m$.

Ex. $\mathbb{Z}/p \xrightarrow{f} \mathbb{Z}/p^2 \xrightarrow{f} \mathbb{Z}/p^3 \rightarrow \dots$

Def. Given a directed poset J and direct system $\{G_\alpha, f_{\alpha\beta}\}$ define

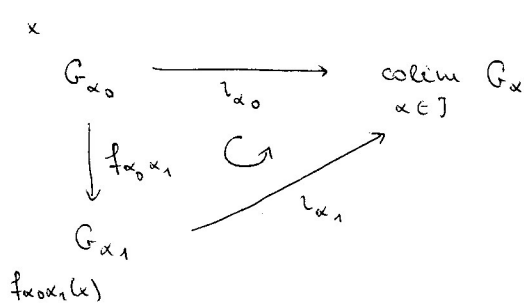
$$\underline{\text{colim}_{\alpha \in J} G_\alpha} := \bigoplus_{\alpha \in J} G_\alpha / (x - f_{\alpha\beta}(x)) \quad \forall x \in G_\alpha, \alpha \leq \beta$$

Note that as usual, we abuse the notation $x \in G_{\alpha_0} \hookrightarrow \bigoplus_{\alpha \in J} G_\alpha$.



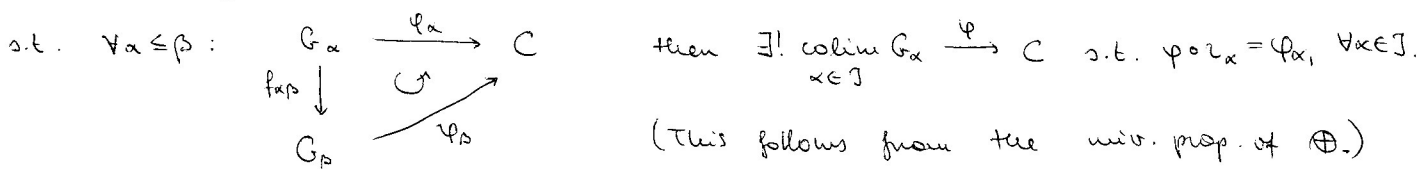
$$\iota_\alpha(x) = [x] = [(x_\beta)] \text{ where } x_\beta = \begin{cases} 0 & \beta \neq \alpha \\ x & \beta = \alpha \end{cases}$$

Properties.
(PO)



$[x] = [f_{\alpha_0 \alpha_1}(x)]$ since we quotient out by their difference.

Universal property: given any ab. gp C with maps $G_\alpha \xrightarrow{\varphi_\alpha} C \quad \forall \alpha \in I$



Alternative description of $\text{colim}_{\alpha \in I} G_\alpha$: (here we really use that I is directed)

$$\text{colim}_{\alpha \in I} G_\alpha \cong \coprod_{\alpha \in I} G_\alpha / \sim \quad \text{where for } a \in G_\alpha, b \in G_\beta \text{ we have}$$

$$a \sim b \iff \exists \gamma \in I: \alpha \leq \gamma, \beta \leq \gamma, f_{\alpha\gamma}(a) = f_{\beta\gamma}(b).$$

Group structure: $a \in G_\alpha, b \in G_\beta: [a] + [b] = [f_{\alpha\gamma}(a) + f_{\beta\gamma}(b)]$ for some $\alpha \leq \gamma, \beta \leq \gamma$.

Cor. $x \in G_\alpha$. Then $0 = [x] \in \text{colim}_{\alpha \in I} G_\alpha$ iff $\exists \beta \geq \alpha: f_{\alpha\beta}(x) = 0 \in G_\beta$.

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow \text{colim}_{n \in \mathbb{N}} A_n$$

Ex. $\text{colim}_{n \in \mathbb{N}} \mathbb{Z}/p^n = \mathbb{Z}/p^\infty = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$ Prüfer group

We generalise this from ab. gps to chain complexes.

Def. A direct system (I -indexed) of cochain complexes is the following

- data.
- $\forall \alpha \in I$. C_α cochain complex
 - $\forall \alpha \leq \beta$: $f_{\alpha\beta}: C_\alpha \rightarrow C_\beta$

such that $f_{\alpha\alpha} = \text{id}_{C_\alpha}$, $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$.

Now $\text{colim}_{\alpha \in J} C_\alpha$ as a cochain complex is defined as:

$$\left(\text{colim}_{\alpha \in J} C_\alpha \right)^n = \text{colim}_{\alpha \in J} C_\alpha^n \quad n \in \mathbb{Z}$$

$$\begin{array}{ccccc} C_\alpha^n & \xrightarrow{d_\alpha^n} & C_\alpha^{n+1} & \xrightarrow{r_\alpha^{n+1}} & \text{colim}_{\alpha \in J} C_\alpha^{n+1} \\ \downarrow f_{\alpha/\beta} & \curvearrowright & \downarrow f_{\alpha/\beta}^{n+1} & \nearrow r_\beta^{n+1} & \\ C_\beta^n & \xrightarrow{d_\beta^n} & C_\beta^{n+1} & & \end{array}$$

Differential: $\exists! d: \text{colim}_{\alpha \in J} C_\alpha^n \xrightarrow{d} \text{colim}_{\alpha \in J} C_\alpha^{n+1}$, $d \circ r_\alpha^n = r_\alpha^{n+1} \circ d$, $d^2 = 0$.

$$\begin{array}{ccc} [c] & \longrightarrow & [d(c)] \end{array}$$

Back to cpt supp chg.

Given a space X , the set of all compact subsets of X is a directed poset:

$$\forall K, K': K \subseteq K \cup K' \supseteq K'$$

For $K \subseteq L$ compact subsets of X : $X \setminus L \subseteq X \setminus K$

$$C^*(X, X \setminus K; \mathbb{R}) \xrightarrow[\substack{i_{KL}^* = i^* \\ (\text{a.o.n.})}]{\quad} C^*(X, X \setminus L; \mathbb{R})$$

Prop. (Exercise) $(C^*(X, X \setminus K; \mathbb{R}), i_{KL}^*)$ is a directed system of cochain cxs.

Prop. 1) $\text{colim}_K C^*(X, X \setminus K; \mathbb{R}) \cong C_c^*(X; \mathbb{R})$
(Exc.)

2) Given a directed system C_α of cochain complexes,

$$H^l \left(\text{colim}_{\alpha \in J} C_\alpha \right) \xleftarrow{\cong} \text{colim}_{\alpha \in J} H^l(C_\alpha).$$

Cor. The canonical map

$$\text{colim}_K H^l(X, X \setminus K; \mathbb{R}) \xrightarrow{\cong} H_c^l(X; \mathbb{R}) \text{ is an isomorphism.}$$

Ex. Compute $H_c^l(\mathbb{R}^n, G) \cong \text{colim}_K H^l(X, X \setminus K, G)$

Prop. (Exercise) $J \subseteq J'$ is afinal if $\forall \alpha \in J \exists \beta \in J': \alpha \in \beta$.

$$G_\beta \xrightarrow{i_\beta} \text{colim}_{\alpha \in J} G_\alpha \text{ induce by the univ prop a unique}$$

$$\text{colim}_{\beta \in J'} G_\beta \longrightarrow \text{colim}_{\alpha \in J} G_\alpha$$

Now in our example, let \mathcal{J} be the poset of balls of radius n .

This is cofinal in all compacts.

$$\Rightarrow H_c^l(\mathbb{R}^n, G) \cong \operatorname{colim}_{m \in \mathbb{N}} H^l(\mathbb{R}^n, \mathbb{R}^n \setminus B(m), G)$$

where $B(m)$ is the ball of radius m .

$$H^l(\mathbb{R}^n, \mathbb{R}^n \setminus B(m), G) \cong \begin{cases} 0 & l \neq n \\ G & l = n \end{cases}$$

$$H_c^l(\mathbb{R}^n, G) = 0 \text{ unless } n = l$$

$$H_c^l(\mathbb{R}^n, \mathbb{R}^n \setminus B(m); G) \longrightarrow H_c^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(m), G)$$

$$\begin{array}{ccc} \cong & & \cong \\ G & \xrightarrow{\text{id}} & G \end{array}$$

$$\operatorname{colim}_{m \in \mathbb{N}} H_c^n(\mathbb{R}^n, \mathbb{R}^n \setminus B(m); G) = \operatorname{colim}_{m \in \mathbb{N}} G \cong G \Rightarrow H_c^n(\mathbb{R}^n; G) \cong G$$

Prop. (Exercise) If a directed poset \mathcal{J} has a largest element $\gamma \in \mathcal{J}$

(i.e. $\forall \alpha \in \mathcal{J}: \alpha \leq \gamma$) then $\operatorname{colim}_{\alpha \in \mathcal{J}} G_\alpha \cong G_\gamma$.

$$H_c^*(X; \mathbb{R}) = \operatorname{colim}_K H^*(X, X \setminus K; \mathbb{R}) \cong H^*(X, X \setminus X; \mathbb{R}) = H^*(X; \mathbb{R})$$

↑
for X cpt.

So compactly supported chg. for cpts is just the usual chg.

30.05.2018

Let M be an \mathbb{R} -orientable n -manifold. Then $\forall K \subset M$ cpt. subset we have

$$\cap: H_n(M, M \setminus K; \mathbb{R}) \otimes_{\mathbb{R}} H^l(M, M \setminus K; \mathbb{R}) \longrightarrow H_{n-l}(M; \mathbb{R})$$

Note that $\{\mu_x \in H_n(M, M \setminus \{x\}; \mathbb{R}) \mid x \in M\}$ is an orientation of M (def.) and

by a previous Lemma $\exists! \mu_K \in H_n(M, M \setminus K; \mathbb{R})$ s.t. $\operatorname{res}_x(\mu_K) = \mu_x \quad \forall x \in K$.

If $K \subseteq L \subseteq M$ are cpt subsets of M then $K \xrightarrow{r} L$ induces

$$H_n(M, M \setminus L; \mathbb{R}) \xrightarrow{r_*} H_n(M, M \setminus K; \mathbb{R}).$$

The uniqueness in the last statement implies $r_*(\mu_L) = \mu_K$ since they restrict to the same thing.

"I don't want to mix up some formulas with actual mathematics."

Recall: $f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$

Claim:
$$\begin{array}{ccc} H^l(M, M \setminus K; \mathbb{R}) & \xrightarrow{\mu_K \cap -} & H_{n-l}(M; \mathbb{R}) \\ \downarrow \iota^* & \nearrow G & \\ H^l(M, M \setminus L; \mathbb{R}) & \xrightarrow{\mu_L \cap -} & \end{array}$$

Using the formula above:

$$\underbrace{\iota_*(\mu_L)}_{\mu_K} \cap [\varphi] = \text{id}_*(\mu_L \cap \iota^*[\varphi]) = \mu_L \cap \iota^*[\varphi] \Rightarrow \text{Claim.}$$

Thus we have a map D_M by the univ. prop:

$$\begin{array}{ccc} \text{colim}_{K \subset M} H^l(M, M \setminus K; \mathbb{R}) & \xrightarrow{D_M} & H_{n-l}(M; \mathbb{R}) \\ \downarrow \iota_K & \nearrow \mu_K \cap - & \\ H^l(M, M \setminus K; \mathbb{R}) & & \end{array}$$

Thm. (Poincaré duality for not necessarily opt manifolds)

For \mathbb{R} -orientable n -manifolds M D_M is an isomorphism.

This clearly generalises the previous version of P. duality.

Prop. Let M be an \mathbb{R} -orientable n -dim mf. and suppose $M = U \cup V$ where U, V are open. Then there is a commutative diagram:

$$\begin{array}{ccccccc} \dots \rightarrow H_c^l(U \cap V) & \xrightarrow{(\iota_{1*}, \iota_{2*})} & H_c^l(U) \oplus H_c^l(V) & \xrightarrow{(\iota_{1*}, \iota_{2*})} & H_c^l(U \cup V) & \xrightarrow{(-1)^{l+1} \partial} & H_c^{l+1}(U \cap V) \rightarrow \dots \\ \downarrow D_{U \cap V} & & \downarrow D_U \oplus D_V & & \downarrow D_{U \cup V} & & \downarrow D_{U \cap V} \\ \dots \rightarrow H_{n-l}(U \cap V) & \xrightarrow{(\iota_{1*}, \iota_{2*})} & H_{n-l}(U) \oplus H_{n-l}(V) & \xrightarrow{(\iota_{1*}, \iota_{2*})} & H_{n-l}(U \cup V) & \xrightarrow{\partial} & H_{n-l-1}(U \cap V) \rightarrow \dots \end{array}$$

where both rows are exact, and the bottom row is the MV sequence in homology.

$$\begin{aligned} \iota_1: U \cap V &\hookrightarrow U, & \iota_2: U \cap V &\hookrightarrow V, \\ i_U: U &\hookrightarrow U \cup V, & i_V: V &\hookrightarrow U \cup V. \end{aligned}$$

Pf. For a space X and $U \xrightarrow{\iota} X$ open we have

$$\begin{array}{ccc} H_c^*(U) & \xrightarrow{\iota^*} & H_c^*(X) \\ \cong & & \cong \\ \operatorname{colim}_{K \subseteq U} H^*(U, U \setminus K) & & \operatorname{colim}_{X \subseteq U} H^*(X, X \setminus K) \end{array}$$

Excision $\rightarrow H^*(X, X \setminus K) \xrightarrow{\cong} H^*(U, U \setminus K)$ if $K \subseteq U$.

$$\Rightarrow H_c^*(U) \cong \operatorname{colim}_{K \subseteq U} H^*(X, X \setminus K) \xrightarrow{\iota^*} \operatorname{colim}_{K \subseteq X} H^*(X, X \setminus K)$$

Moreover if $U \xrightarrow{i} V \xrightarrow{j} X$ then $(j \circ i)_* = j_* \circ i_*$, i.e. we have functoriality.

Let $K \subseteq U$, $L \subseteq V$ be cpt. $\Rightarrow K \cup L \subseteq U \cup V = M$, $K \cap L \subseteq U \cap V$.

$$H_c^l(U) = \operatorname{colim}_{K \subseteq U} H^l(U, U \setminus K) \cong \operatorname{colim}_{K \subseteq U} H^l(M, M \setminus K)$$

$$H_c^l(V) = \operatorname{colim}_{L \subseteq V} H^l(V, V \setminus L) \cong \operatorname{colim}_{L \subseteq V} H^l(M, M \setminus L)$$

$M \setminus (K \cup L) \subseteq M \setminus K$ and $M \setminus (K \cup L) \subseteq M \setminus L$. We get induced maps

$$H^l(M, M \setminus K) \xrightarrow{z_K} H^l(M, M \setminus (K \cup L))$$

$$\parallel \\ H^l(M, M \setminus L) \xrightarrow{z_L} H^l(M, M \setminus (K \cup L))$$

$$H^l(M, M \setminus K) \oplus H^l(M, M \setminus L) \xrightarrow{(z_K, -z_L)} H^l(M, M \setminus (K \cup L))$$

Pass to colimits along K and L :

$$H_c^l(U) \oplus H_c^l(V) = \operatorname{colim}_{K \subseteq U} H^l(M, M \setminus K) \oplus \operatorname{colim}_{L \subseteq V} H^l(M, M \setminus L)$$

$$\searrow \qquad \downarrow \\ H_c^l(M) \cong \operatorname{colim}_{K \subseteq U, L \subseteq V} H^l(M, M \setminus (K \cup L))$$

using cofinality.

$$H^l(M, M \setminus (K \cap L)) \longrightarrow H^l(M, M \setminus K) \oplus H^l(M, M \setminus L)$$

Pass to colim along K, L , colim commutes with \oplus

$$\operatorname{colim}_{K \subseteq U, L \subseteq V} H^l(M, M \setminus (K \cap L)) \longrightarrow \operatorname{colim}_{K \subseteq U} H^l(M, M \setminus K) \oplus \operatorname{colim}_{L \subseteq V} H^l(M, M \setminus L)$$

$$\parallel \qquad \parallel \\ H_c^l(U \cap V) \longrightarrow H_c^l(U) \oplus H_c^l(V)$$

Now we see that the maps involved in the Prop are colimits of certain maps for K, L . Thus we may deal with these maps and then pass to colim.

CCM opt., $C \subseteq U$ open, $\mu_C \in H_n(M, M \setminus C)$ (directed colim is an exact functor)

$$H_n(M, M \setminus C) \cong H_n(U, U \setminus C) \ni \mu_C \text{ is a singular chain in } U$$

from now on we identify these two groups

Then it suffices to show that the following commutes:

$$\begin{array}{ccccccc}
 \dots \rightarrow H^l(M, M \setminus (K \cup L)) & \xrightarrow{(+)} & H^l(M, M \setminus K) \oplus H^l(M, M \setminus L) & \xrightarrow{(+)} & H^l(M, M \setminus (K \cup L)) & \xrightarrow{(-)^{l+1} \delta} & H^{l+1}(M, M \setminus (K \cup L)) \rightarrow \dots \\
 \downarrow \cong \text{exc.} & & \cong \downarrow \text{exc} \oplus (-\text{exc}) & & \downarrow \mu_{K \cup L} \cap - & \text{(\#)} & \cong \downarrow \text{exc.} \\
 H^l(U \cap V, (U \cap V) \setminus (K \cup L)) & & H^l(U, U \setminus K) \oplus H^l(V, V \setminus L) & & \downarrow \mu_{K \cup L} \cap - & & H^{l+1}(U \cap V, (U \cap V) \setminus (K \cup L)) \\
 \downarrow \mu_{K \cup L} \cap - & & \downarrow (\mu_U \cap -) \oplus (\mu_V \cap -) & & & & \downarrow \mu_{K \cup L} \cap - \\
 \dots \rightarrow H_{n-l}(U \cap V) & \xrightarrow{(+)} & H_{n-l}(U) \oplus H_{n-l}(V) & \xrightarrow{(+)} & H_{n-l}(U \cup V) & \xrightarrow{\partial} & H_{n-l-1}(U \cap V) \rightarrow \dots
 \end{array}$$

The top row is the relative MV sequence for $(M, A = M \setminus K, B = M \setminus L)$.

The first two squares are easily seen to commute: this can be decided on the cycle level. The painful part, as always, is the one involving the boundary homomorphisms. (#)

$$[\varphi] \in H^l(M, M \setminus (K \cup L))$$

$$\mu_{K \cup L} \cap \text{exc} \circ \delta [\varphi] = ? \quad \leftarrow \text{this is the hard part}$$

$$\begin{array}{ccccccc}
 0 \rightarrow C^*(M, A+B) & \xrightarrow{\psi} & C^*(M, A) \oplus C^*(M, B) & \rightarrow & C^*(M, A \cap B) & \rightarrow 0 & \text{sh. ex. 29} \\
 \downarrow \psi & & \downarrow \psi_A \quad \downarrow \psi_B & & \downarrow \psi & & \\
 \text{Recall: } \psi & \text{if } \psi|_{C_*(A)} = 0 = \psi|_{C_*(B)} & \text{(but } \psi|_{C_*(A \cup B)} \text{ need not be 0)} & & & &
 \end{array}$$

$$H^l(C^*(M, A+B)) \cong_{\text{exc}} H^l(C^*(M, A \cup B)) = H^l(M, A \cup B)$$

For $\varphi \in C^*(M, A \cap B)$ find $\varphi_A \in C^*(M, A), \varphi_B \in C^*(M, B)$ s.t. $\varphi = \varphi_A - \varphi_B$.

$$(d\varphi_A, d\varphi_B) \in C^*(M, A) \oplus C^*(M, B)$$

$$d\varphi_A - d\varphi_B = d\varphi = 0 \text{ as } \varphi \text{ is a cocycle} \Rightarrow d\varphi_A = d\varphi_B$$

$$\Rightarrow d\varphi_A \text{ gets sent to } (d\varphi_A, d\varphi_A) = (d\varphi_A, d\varphi_B) \Rightarrow [d\varphi_A] = \delta[\varphi].$$

$\Rightarrow \delta[\varphi] = [d\varphi_A + d\psi]$ in $H^*(M, A \cup B)$ where ψ is a cocycle with

$$\psi|_{C^*(M,A)} = 0, \psi|_{C^*(M,B)} = 0.$$

There is a canonical morphism $H^*(M, A \cup B) \xrightarrow{\cong} H^*(M, A+B)$.

04.06.2018

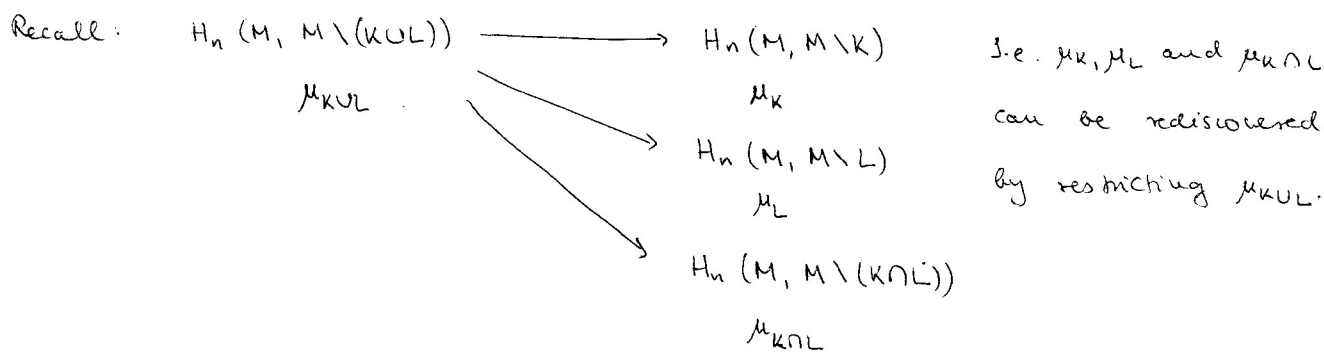
A priori $[d\varphi_A]$ sits in $H^*(M, A+B)$, this is why we need this error term $d\psi$.

$$M = (U \setminus L) \cup (U \cap V) \cup (V \setminus K)$$

By using the lemma about small simplices (Top I): there is a representative $\mu_{KUL} \in H_n(M, M \setminus (K \cup L))$ of the form

$$\alpha_{U \setminus L} + \alpha_{U \cap V} + \alpha_{V \setminus K}$$

where $\alpha_{U \setminus L}$, $\alpha_{U \cap V}$ and $\alpha_{V \setminus K}$ are singular chains in $U \setminus L$, $U \cap V$, $V \setminus K$, respectively.



$\alpha_{U \cap V}$ is a rep. for $\mu_{K \cap L}$: $\alpha_{U \setminus L}$ and $\alpha_{V \setminus K}$ are already zero in $C_n(M, M \setminus (K \cap L))$

Similarly: $\alpha_{U \setminus L} + \alpha_{U \cap V}$ is a rep. for μ_K ,

$\alpha_{U \cap V} + \alpha_{V \setminus K}$ is a rep. for μ_L .

(Note that $[\alpha_{U \cap V}] \in H_n(U \cap V, U \cap V \setminus (K \cap L)) \xrightarrow{\cong} H_n(M, M \setminus (K \cap L))$.)

$\Rightarrow \mu_{K \cap L} \cap \text{exc} \circ \delta[\varphi]$ is represented by $\alpha_{U \cap V} \cap (d\varphi_A + d\psi)$.

$\alpha_{U \cap V} \cap (d\varphi_A + d\psi)$ is a cycle in $U \cap V$.

We claim that $\alpha_{U \cap V} + d\psi$ is a boundary (and hence we can get rid of it)

$$d(\alpha_{U \cap V} \cap \psi) = (-1)^{\deg \psi} \left(\underbrace{d\alpha_{U \cap V} \cap \psi}_{\text{wts} = 0} - \alpha_{U \cap V} \cap d\psi \right)$$

$$d(\alpha_{U \cap V} + \alpha_{U \setminus L} + \alpha_{V \setminus K}) = 0 \text{ in } C_n(M, M \setminus (K \cup L))$$

$$\Rightarrow d\alpha_{U \cap V} \cap \psi = \underbrace{-d\alpha_{U \setminus L} \cap \psi}_0 - \underbrace{d\alpha_{V \setminus K} \cap \psi}_0$$

recall: $\psi|_{C^*(M,K)} = 0, \psi|_{C^*(M,L)} = 0$.

$$\Rightarrow \alpha_{U \cap V} \cap d\psi = (-1)^{\deg \psi} d(\alpha_{U \cap V} \cap \psi).$$

(Note that this step is missing in Hatcher.)

So from now on we work with $[\alpha_{UV} \cap d\varphi_A]$.

$$d(\alpha_{UV} \cap \varphi_A) = (-1)^l \cdot (d\alpha_{UV} \cap \varphi_A - \alpha_{UV} \cap d\varphi_A)$$

So we have

$$[\alpha_{UV} \cap d\varphi_A] = \underbrace{[d\alpha_{UV} \cap \varphi_A]}_{= \mu_{K \cap L} \cap \text{exc} \circ \delta[\varphi]} \quad \text{in } H_{n-l-1}(U \cap V), \quad (*)$$

Now we relate the homological and cohomological side to each other.

$$\begin{aligned} \partial(\mu_{K \cap L} \cap [\varphi]) &= \partial(\underbrace{\alpha_{U \cap L} \cap \varphi}_{\text{chain in } U} + \underbrace{\alpha_{U \cap V} \cap \varphi + \alpha_{V \cap K} \cap \varphi}_{\text{chain in } V}) = \\ &= [d(\alpha_{U \cap L} \cap \varphi)]_{d\varphi=0} = (-1)^l [d\alpha_{U \cap L} \cap \varphi] = \varphi = \varphi_A - \varphi_B \\ &= (-1)^l [d\alpha_{U \cap L} \cap \varphi_A] = \end{aligned}$$

$\alpha_{UV} + \alpha_{UL}$ represents $\mu_K \Rightarrow d(\alpha_{UV} + \alpha_{UL})$ is a chain in $M \setminus K$

$$= (-1)^l [-d\alpha_{UV} \cap \varphi_A] = (-1)^{l+1} [d\alpha_{UV} \cap \varphi_A],$$

which is the same as in (*) and the sign is as stated in the assertion. \square

Now we finally prove Poincaré duality (Hatcher, Milnor - Stasheff):

$$D_M: H_c^l(M; \mathbb{R}) \longrightarrow H_{n-l}(M; \mathbb{R})$$

is an iso for any \mathbb{R} -orientable n -manifold M .

PROOF: Case 1. $M = \mathbb{R}^n$

$$D: H_c^l(\mathbb{R}^n) \longrightarrow H_{n-l}(\mathbb{R}^n)$$

If $l \neq n$, both sides are zero.

$$\text{If } l = n: \mathbb{R} \cong H_c^n(\mathbb{R}^n) \xrightarrow{D} H_0(\mathbb{R}^n) \cong \mathbb{R}$$

$$\begin{array}{ccc} \uparrow \cong & \subset & \nearrow \\ H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) & & \mu_B \cap - \end{array}$$

This holds for any closed ball $B \subset \mathbb{R}^n$.

\Rightarrow Sts: $\mu_B \cap -$ is an iso.

The Kronecker product $H^0(\mathbb{R}^n; \mathbb{R}) \otimes_R H_0(\mathbb{R}^n; \mathbb{R}) \xrightarrow{\langle -, - \rangle} \mathbb{R}$ is non-degenerate.

$$\Leftrightarrow \mathbb{R} \otimes_R \mathbb{R} \longrightarrow \mathbb{R} \quad \text{nd} \quad \Leftrightarrow H^0(\mathbb{R}^n; \mathbb{R}) \xrightarrow{\cong} \text{Hom}_R(H_0(\mathbb{R}^n; \mathbb{R}), \mathbb{R})$$

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{R}) \xrightarrow[\cong]{\text{UCT}} \text{Hom}_R(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{R}), \mathbb{R})$$

μ_B

$\exists! x \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B; \mathbb{R})$ which is dual to $\mu_B \Leftrightarrow \langle x, \mu_B \rangle = 1$

Sts $\mu_B \cap x$ is a generator: $\langle 1, \mu_B \cap x \rangle = \langle x, \mu_B \rangle = 1 \Rightarrow \mu_B \cap x$ is a generator ✓

Case 2. $M = \bigcup_{\alpha \in L} U_\alpha$ where the U_α are open and PD holds for all U_α , and $\alpha \leq \beta \Rightarrow U_\alpha \subseteq U_\beta$.

$$\begin{array}{ccc} H_c^l(M) & \xrightarrow{D_M} & H_{n-l}(M) \\ \uparrow & \circlearrowleft & \uparrow \\ H_c^l(U_\alpha) & \xrightarrow[\cong]{D_{U_\alpha}} & H_{n-l}(U_\alpha) \quad \forall \alpha \in L \end{array}$$

This diagram commutes. follows from the defs of the arrows.

Going to directed colims along $\alpha \in L$.

$$\begin{array}{ccc} H_c^l(M) & \xrightarrow{D_M} & H_{n-l}(M) \\ \uparrow & \circlearrowleft & \uparrow \\ \text{colim}_{\alpha \in L} H_c^l(U_\alpha) & \xrightarrow[\cong]{\text{colim}_{\alpha \in L} D_{U_\alpha}} & \text{colim}_{\alpha \in L} H_{n-l}(U_\alpha) \end{array}$$

L totally ordered \Rightarrow every cpt in M is contained in some U_α
 \Rightarrow the vertical maps are isos
 $\Rightarrow D_M$ is an iso ✓

Case 3. $M = U \cup V$, PD holds for the open sets U, V and $U \cap V$

Follows from the 5-lemma and the diagram from before involving duality maps (p 63)

Case 4. $M \subseteq \mathbb{R}^n$ any open subset.

If M is convex: $M \cong \mathbb{R}^n$, use Case 1. ✓

Any open can be written as a countable union $M = V_1 \cup V_2 \cup \dots$

where all the V_i are convex.

$$M = \bigcup_{i=0}^{\infty} \left(\bigcup_{k=1}^i V_k \right)$$

Case 3 \Rightarrow PD holds for $\bigcup_{k=1}^i V_k$ by induction

Case 2 \Rightarrow PD holds for M \checkmark

Case 5. M is arbitrary.

Consider the poset of open subsets of M . This is inductive, i.e. satisfies the condition of Zorn's lemma by Case 2.

Zorn's lemma $\Rightarrow \exists$ maximal open U satisfying PD. Goal: $U = M$.

$\exists U \subsetneq M \Rightarrow \exists x \in M \setminus U$. Take $V \subseteq M$ to be an open nbhd of x s.t. $V \cong \mathbb{R}^n$.

Both U and V satisfy PD (Case 4) $\Rightarrow U \cup V$ satisfies PD (Case 3), $U \subsetneq U \cup V$ \downarrow \square

06.06.2018

Let M be a compact n -dimensional manifold.

$\forall H_i(M; \mathbb{Z})$ is finitely generated because $\exists N$ s.t. there is an embedding $M \hookrightarrow \mathbb{R}^N$ with M being a neighbourhood retract, and hence M has homology groups of a CW complex. (This has been mentioned before; we will treat it as a black box. The proof is mostly point set topology.)

Prop. M cpt. odd dimensional manifold $\Rightarrow \chi(M) = 0$.

From the previous observation it follows that $H_i(M; \mathbb{Z}) \cong \mathbb{Z}^{r_i} \oplus \bigoplus_j \mathbb{Z}/m_j$ (fund. thm. of fin. gen. ab. grps.)
 where $r_i =: \text{rk } H_i(M; \mathbb{Z}) = \dim_{\mathbb{Q}} (H_i(M; \mathbb{Z}) \otimes \mathbb{Q})$. This allows us to define

Def. $\chi(M) := \sum_{i=0}^{+\infty} (-1)^i \text{rk } H_i(M; \mathbb{Z}) = \sum_{i=0}^n (-1)^i \text{rk } H_i(M; \mathbb{Z})$. Euler characteristic

This agrees with our previous definition of χ for CW complexes.

Ex. $\chi(S^2) = \text{rk } H_0 + \text{rk } H_2 = 2$

Pf of Prop. Case 1. M is orientable.

$\Rightarrow \text{rk } H_i(M; \mathbb{Z}) = \text{rk } H^{n-i}(M; \mathbb{Z})$ by PD

UCT $\Rightarrow H^{n-i}(M; \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(H_{n-i}(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^1(H_{n-i-1}(M; \mathbb{Z}), \mathbb{Z})$

In general, for a finitely generated A the ext group $\text{Ext}_{\mathbb{Z}}^1(A, \mathbb{Z})$

is always torsion (and all higher ext groups vanish), because

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}^r, A) = 0 \quad \text{and} \quad \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, B) \cong B/mB.$$

$\Rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-i-1}(M; \mathbb{Z}), \mathbb{Z})$ is torsion \Rightarrow has $\text{rk} = 0$.

$$\Rightarrow \chi(M) = \sum_{i=0}^n (-1)^i \text{rk } H_i(M; \mathbb{Z}) = \text{rk } H_0 - \text{rk } H_1 + \dots + \text{rk } H_{n-1} - \text{rk } H_n = 0$$

Case 2. M is not orientable.

M has PD with \mathbb{F}_2 -coefficients \Rightarrow we can give a similar argument, even easier because $\text{Ext}_{\mathbb{F}_2}^1 = 0$

$$\Rightarrow \underbrace{\chi(M, \mathbb{F}_2)}_{\substack{\text{mod } 2 \\ \text{Euler characteristic}}} := \sum_{i=0}^n (-1)^i \dim H_i(M, \mathbb{F}_2) = 0$$

Now we need to return to $\chi(M)$: we will show

$$\sum_{i=0}^n (-1)^i \dim H_i(M; \mathbb{F}_2) = \sum_{i=0}^n (-1)^i \text{rk } H_i(M; \mathbb{Z})$$

$$\text{UCT} \Rightarrow H^i(M; \mathbb{F}_2) \cong \text{Hom}_{\mathbb{F}_2}(H_i(M; \mathbb{F}_2), \mathbb{F}_2) \cong H_i(M; \mathbb{F}_2)^{\vee} \quad \text{where } \vee \text{ denotes dual}$$

$$\Rightarrow H^i(M; \mathbb{F}_2) \cong H_i(M; \mathbb{F}_2)$$

(Throughout the whole proof, Patchkoria keeps making remarks about the iss involved: namely that sometimes they are not canonical (like at the splitting) or that they may be complicated and the only reason we may neglect this is that we are only interested in the ranks.)

$$H_i(M; \mathbb{Z}) \cong \mathbb{Z}^{r_i} \oplus \bigoplus_{l=1}^{N_i} \mathbb{Z}/m_l \oplus \bigoplus_{j=1}^{K_i} \mathbb{Z}/s_j \quad \text{where } 2|m_l, 2 \nmid s_j$$

$$\begin{aligned} \Rightarrow \dim H^i(M; \mathbb{F}_2) &\stackrel{\text{UCT}}{=} \dim \text{Hom}_{\mathbb{Z}}(H_i(M; \mathbb{Z}), \mathbb{F}_2) + \dim \text{Ext}_{\mathbb{Z}}^1(H_{i-1}(M; \mathbb{Z}), \mathbb{F}_2) = \\ &= r_i + N_i + N_{i-1} \end{aligned}$$

$$\begin{aligned} \underbrace{\sum_{i=0}^n (-1)^i \dim H_i(M; \mathbb{F}_2)}_{0 = \chi(M, \mathbb{F}_2)} &= \sum_{i=0}^n (-1)^i \dim H^i(M; \mathbb{F}_2) \\ &= \sum_{i=0}^n (-1)^i (r_i + N_i + N_{i-1}) \\ &= \sum_{i=0}^n (-1)^i r_i + \underbrace{\sum_{i=0}^n (-1)^i (N_i + N_{i-1})}_{N_0 - (N_1 + N_0) + \dots + (N_{n-1} + N_{n-2}) - (N_n + N_{n-1})} \\ &= \chi(M) + N_n \end{aligned}$$

$H_n(M; \mathbb{Z})$ is torsion free M is not orientable $\Rightarrow N_n = 0 \Rightarrow \chi(M) = 0$.

Now let M be an n -dimensional, \mathbb{R} -orientable, not necessarily connected manifold, $[M] \in H_n(M; \mathbb{R})$ its fundamental class in the following sense.

If $M = \bigsqcup_{\alpha} M_{\alpha}$, $H_n(M; \mathbb{R}) = \bigoplus_{\alpha} \underbrace{H_n(M_{\alpha}; \mathbb{R})}_{\cong \mathbb{R}} \cong \bigoplus_{\alpha} \mathbb{R}$,

$[M] = ([M_{\alpha}])_{\alpha} \in H_n(M; \mathbb{R})$ where $[M_{\alpha}]$ is the fund class of the connected component $[M_{\alpha}]$.

$$\begin{array}{ccc} H^k(M; \mathbb{R}) \otimes_{\mathbb{R}} H^{n-k}(M; \mathbb{R}) & \xrightarrow{\cup_M} & \mathbb{R} \\ \varphi \otimes \psi & \xrightarrow{\quad} & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

intersection product

Recall that $\varphi \cup \psi \in H^{k+n-k}(M; \mathbb{R}) = H^n(M; \mathbb{R})$ and

$$H^n(M; \mathbb{R}) \otimes_{\mathbb{R}} H_n(M; \mathbb{R}) \xrightarrow{\langle -, - \rangle} \mathbb{R}.$$

One may check that \cup_M is well-defined on the tensor product.

We are only interested in the cases $\mathbb{R} = \mathbb{Z}$ and $\mathbb{R} = \text{field}$.

For $\mathbb{R} = \mathbb{Z}$: $H^k(M; \mathbb{Z})_{\text{free}} := H^k(M; \mathbb{Z}) / \text{torsion}$

$$\begin{array}{ccc} H^k(M; \mathbb{Z}) \otimes_{\mathbb{Z}} H^{n-k}(M; \mathbb{Z}) & \xrightarrow{\cup_M} & \mathbb{Z} \\ \downarrow & \nearrow \exists! \cup_M & \\ H^k(M; \mathbb{Z})_{\text{free}} \otimes H^{n-k}(M; \mathbb{Z})_{\text{free}} & & \end{array}$$

We want to work with the free part because then we may use the UCT effectively; in the torsion, things just die.

Prop. Let M be an R -orientable compact n -mf. Then

$$1) U_M^*: H^{n-k}(M; R) \xrightarrow{\cong} \text{Hom}_R(H^k(M; R), R) \text{ if } R \text{ is a field.}$$

$$2) U_M^*: H^{n-k}(M; \mathbb{Z})_{\text{free}} \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(H^k(M; \mathbb{Z})_{\text{free}}, \mathbb{Z}).$$

Ret. In Topology I we have shown that $\chi(M)$ is independent from the choice of the field for CW complexes, but this result of course could not be applied for the general class of top. manifolds.

$$\begin{array}{ccccc} \text{Pf. } H^{n-k}(M; R) & \xrightarrow{h} & \text{Hom}_R(H_{n-k}(M; R), R) & \xrightarrow{D^*} & \text{Hom}_R(H^k(M; R), R) \\ \psi & \longmapsto & \langle \psi, - \rangle & \longmapsto & \langle \psi, [M] \cap - \rangle \end{array}$$

since the duality map $H^k(M; R) \xrightarrow{D} H_{n-k}(M; R)$ is given by $[M] \cap -$.

$$\varphi \mapsto \langle \psi, [M] \cap \varphi \rangle = \langle \varphi \cup \psi, [M] \rangle \text{ by the cap-cup formula}$$

$$U_M^* := D^* \circ h.$$

1) If R is a field: h is an iso by UCT, D is an iso by PD
 $\Rightarrow U_M^*$ is an iso.

$$\begin{array}{ccccc} 2) H^{n-k}(M; \mathbb{Z})_{\text{free}} & \xrightarrow[\cong]{h} & \text{Hom}_{\mathbb{Z}}(H_{n-k}(M; \mathbb{Z}), \mathbb{Z}) & \xrightarrow[\cong]{D^*} & \text{Hom}_{\mathbb{Z}}(H^k(M; \mathbb{Z}), \mathbb{Z}) \\ & & & & \parallel \\ & & & & \text{Hom}_{\mathbb{Z}}(H^k(M; \mathbb{Z})_{\text{free}}, \mathbb{Z}) \end{array}$$

U_M^*

Cor. M cpt ori n -dim connected mf. Suppose $\alpha \in H^k(M; \mathbb{Z})$ is of infinite order and α is not a proper multiple of any element, i.e. if $\exists m \in \mathbb{Z}, \gamma \in H^k(M; \mathbb{Z})$ s.t. $\alpha = m\gamma$ then $m = \pm 1$.

Then $\exists \beta \in H^{n-k}(M; \mathbb{Z})$ such that $\alpha \cup \beta \in H^n(M; \mathbb{Z}) \cong \mathbb{Z}$ is a generator

Note that β is only unique up to sign and torsion.

$$\text{Pf. } H^k(M; \mathbb{Z}) \text{ is finitely generated. } \Rightarrow H^k(M; \mathbb{Z}) \cong \underbrace{\mathbb{Z}\langle \alpha \rangle}_{\mathbb{Z} \text{ generated by } \alpha} \oplus B$$

$$\Rightarrow \exists \varphi: H^k(M; \mathbb{Z}) \longrightarrow \mathbb{Z} \text{ s.t. } \varphi(\alpha) = 1.$$

By the Prop: $\exists \beta \in H^{n-1}(M; \mathbb{Z})$ s.t. $U_M^*(\beta) = \varphi$.

$$U_M^*(\beta) = \langle -U\beta, [M] \rangle = \varphi(-) \Rightarrow \langle \alpha U\beta, [M] \rangle = \varphi(\alpha) = 1.$$

$H^n(M; \mathbb{Z}) \xrightarrow{\cong} \text{Hom}(H_n(M; \mathbb{Z}), \mathbb{Z})$ Recall that the torsion of $H_{n-1}(M; \mathbb{Z})$ is 0.

$$\alpha U\beta \longmapsto \underbrace{\langle \alpha U\beta, - \rangle}$$

this is a generator $\Rightarrow \alpha U\beta$ is a generator

Let $F = \mathbb{R}, \mathbb{C}$ or \mathbb{H} (but not \mathbb{D})

$$FP^n := F^{n+1} \setminus \{0\} / \sim \quad \text{projective space}$$

$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \Leftrightarrow \exists \lambda \in F^* \text{ s.t. } \lambda x_i = y_i \quad \forall i = 0, \dots, n$

The topology is the quotient topology from $F^{n+1} \setminus \{0\}$

FP^n are n -dimensional cpt connected manifolds

$$U_i \subseteq FP^n \quad U_i = \{[x_0, \dots, x_n] \in FP^n \mid x_i \neq 0\}, \quad F^n \xrightarrow{\cong} U_i$$

$$(x_1, \dots, x_n) \longmapsto [x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n]$$

$$FP^n = U_0 \cup \dots \cup U_n$$

Let $S(F^{n+1})$ be the unit sphere in F^{n+1} . Then the projection

$$S(F^{n+1}) \longrightarrow FP^n \text{ is surjective: } [x_0, \dots, x_n] = \left[\frac{x_0}{\|x\|}, \dots, \frac{x_n}{\|x\|} \right] \text{ where } \|x\| = \sqrt{\sum x_i^2}$$

$F = \mathbb{R}$: RP^n is a CW complex

$$\begin{array}{ccc} S^n & \longrightarrow & RP^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & RP^{n+1} \end{array}$$

$$H_n(RP^n; \mathbb{Z}) = \begin{cases} 0 & n \geq 2, 2 \nmid n \\ \mathbb{Z} & 2 \mid n \end{cases}$$

$\Rightarrow RP^n$ is orientable if $2 \nmid n$ and non-orientable if $2 \mid n$.

$$H_i(RP^n; \mathbb{F}_2) = \begin{cases} 0 & i > n \\ \mathbb{F}_2 & 0 \leq i \leq n \end{cases}$$

$$\Rightarrow H^i(RP^n; \mathbb{F}_2) = \begin{cases} 0 & i > n \\ \mathbb{F}_2 & 0 \leq i \leq n \end{cases}$$

F = C: $S^{2n+1} \longrightarrow \mathbb{C}P^n$ CW complex with no odd dimensional cells

$$\begin{array}{ccc} \downarrow & & \downarrow \\ D^{2n+2} & \longrightarrow & \mathbb{C}P^{n+1} \end{array}$$

$$H_i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, 2|i \\ 0 & \text{else} \end{cases}$$

$$\text{UCT} \Rightarrow H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, 2|i \\ 0 & \text{else} \end{cases}$$

F = H: $S^{4n+3} \longrightarrow \mathbb{H}P^n$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ D^{4n+4} & \longrightarrow & \mathbb{H}P^{n+1} \end{array}$$

$$H_i(\mathbb{H}P^n; \mathbb{Z}) \cong H^i(\mathbb{H}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 4n, 4|i \\ 0 & \text{else} \end{cases}$$

What about \mathbb{O} ? The projective spaces $\mathbb{O}P^1 = S^8$ and $\mathbb{O}P^2$ (Cayley plane) can be defined. The relation won't be an eq. relation. Of course it can be made into one but it won't yield a manifold.

Next goal: compute the cohomology rings using the Cor. (p. 72).